

M-theory superalgebra from the M-5-brane

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Abstract

The algebra of Noether supercharges of the M-5-brane effective action is shown to include both 2-form and 5-form central charges. Surprisingly, only the 5-form charge is entirely due to the Wess-Zumino term because the ‘naive’ algebra is the M-2-brane supertranslation algebra. The full structure of central charges is shown to be directly related to the projector arising in the proof of κ -symmetry of the M-5-brane action. It is also shown to allow ‘mixed’ M-brane configurations preserving 1/2 supersymmetry that include the (non-marginal) M-2-brane/M-5-brane ‘bound state’ as a special case.

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1 Introduction

The most general supertranslation algebra in eleven-dimensional spacetime is spanned by a Majorana spinor supercharge Q_α ($\alpha = 1, \dots, 32$), the 11-momentum P_m , a 2-form central charge Z_{mn} and a 5-form central charge $Y_{m_1 \dots m_5}$ [1, 2]. The only non-vanishing (anti)commutation relation is the anticommutator

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^m)_{\alpha\beta}P_m + \frac{1}{2}(C\Gamma^{mn})_{\alpha\beta}Z_{mn} + \frac{1}{5!}(C\Gamma^{m_1 \dots m_5})_{\alpha\beta}Y_{m_1 \dots m_5}, \quad (1)$$

where Γ^m are (constant) Dirac matrices, $\Gamma^{m_1 \dots m_p}$ ($p = 2, 5$) are their antisymmetrized products, and C is the (real) charge conjugation matrix. We use here the ‘mostly plus’ metric convention for which the Dirac matrices are real in the Majorana representation, which we adopt below.

This superalgebra was called the M-theory superalgebra in [2] because its structure reflects important aspects of what has come to be known as M-theory. For example, setting $m = (0, \underline{m})$ ($\underline{m} = 1, \dots, 10$), the positivity of the left hand side of (1) implies that P^0 satisfies a bound of the type

$$P^0 \geq f(P_{\underline{m}}, Z_{\underline{m}\underline{n}}, Y_{\underline{m}_1 \dots \underline{m}_5}; Z_{0\underline{m}}, Y_{0\underline{m}_1 \dots \underline{m}_4}) \quad (2)$$

for some particular function f . Each of the arguments of this function is a charge carried by one of the ‘basic’ objects of M-theory. For example, the 10-momentum $P_{\underline{m}}$ is associated with the D=11 massless superparticle [3], the 2-form charge $Z_{\underline{m}\underline{n}}$ is associated with the supermembrane [4], or M-2-brane, and (as verified here) both, the 2-form charge and the 5-form charge $Y_{\underline{m}_1 \dots \underline{m}_5}$, are associated with the M-theory fivebrane [5, 6], or M-5-brane. These ‘basic’ objects are also represented by classical solutions of D=11 supergravity [7, 8, 9], and the M-5-brane was originally discovered this way. The time components of Z and Y are charges carried by objects that appear in Kaluza-Klein (KK) vacua. Specifically, $Y_{0\underline{m}_1 \dots \underline{m}_4}$ is carried by the D-6-brane of IIA superstring theory while $Z_{0\underline{m}}$ can be associated with the IIA D-8-brane¹. We shall not have anything further to say here about these ‘KK-type’ charges, as we shall assume an uncompactified D=11 spacetime. Setting them to zero, and making use of the fact that $C = \Gamma^0$ in the Majorana representation, we can rewrite (1) as

$$\{Q, Q\} = P^0(1 + \bar{\Gamma}) \quad (3)$$

where

$$\bar{\Gamma} = (P^0)^{-1} \left[\Gamma^{0\underline{m}} P_{\underline{m}} + \frac{1}{2} \Gamma^{0\underline{m}\underline{n}} Z_{\underline{m}\underline{n}} + \frac{1}{5!} \Gamma^{0\underline{m}_1 \dots \underline{m}_5} Y_{\underline{m}_1 \dots \underline{m}_5} \right]. \quad (4)$$

The bound (2) is now equivalent to the statement that no eigenvalue of $\bar{\Gamma}^2$ can exceed unity. Those choices of central charges for which

$$\bar{\Gamma}^2 = 1 \quad (5)$$

are of particular significance because $\frac{1}{2}(1 + \bar{\Gamma})$ is then a projector, projecting onto the eigenspace of $\bar{\Gamma}$ with eigenvalue 1. Since $\bar{\Gamma}$ has zero trace, the dimension of this eigenspace is half that of the original space. This means that configurations associated with such charge values preserve 1/2 the supersymmetry.

It is easy to see that the condition (5) is solved by the following three choices, in which all charges independent of the one given vanish:

$$\begin{aligned} (i) \quad P_1 &= P^0 \\ (ii) \quad Z_{12} &= P^0 \\ (iii) \quad Y_{12345} &= P^0. \end{aligned}$$

These charges are naturally associated with a wave in the 1-direction, a membrane in the 12-plane and a fivebrane in the 12345-plane, respectively. Of course, these are not the only solutions of (5). There is

¹This interpretation of the time components of the central charges has been noticed independently by C. Hull [10].

the obvious rotational freedom, and also the freedom to consider a combination with both P_1 and Z_{23} non-zero, in which case

$$P^0 = \sqrt{P_1^2 + Z_{23}^2}. \quad (6)$$

These charges have an obvious interpretation as those of a membrane in the 23-plane boosted in the 1-direction. Discounting the freedom to rotate and boost, the configurations (i) and (ii) are the only solutions of (5) with $Y = 0$.

In contrast, when $Y \neq 0$ there are many more possibilities than just (iii). For example, (5) is also solved by

$$Z_{12} = \zeta, \quad Y_{12345} = \sqrt{(P^0)^2 - \zeta^2}, \quad (7)$$

where ζ is (classically) arbitrary apart from the constraint $|\zeta| \leq P^0$. These charges can be interpreted as those of a (non-marginal) M-2-brane/M-5-brane bound state; the corresponding solution of D=11 supergravity preserving 1/2 supersymmetry was given in [11]. A still more general ‘mixed’ M-brane solution of (5) is

$$Z_{12} = \zeta_1, \quad Z_{34} = \zeta_2, \quad P_5 = \zeta_1 \zeta_2, \quad Y_{12345} = \sqrt{(P^0)^2 - \zeta_1^2 - \zeta_2^2 - \zeta_1^2 \zeta_2^2}. \quad (8)$$

This case, which we shall discuss further at the conclusion of this paper, presumably corresponds to a bound state of an M-5-brane with two boosted intersecting M-2-branes, although the associated stationary solution of D=11 supergravity is not yet known. Note that if we were to ‘strip off’ the fivebrane covering the two membranes (i.e. put $Y = 0$) we would get a boosted configuration of two M-2-branes intersecting at a point, which preserves 1/4 supersymmetry. Similarly, if we were to omit the momentum then we would have a static configuration of two intersecting membranes in a fivebrane preserving 1/4 supersymmetry. What this shows is that the addition of a brane (or a wave) to a configuration preserving supersymmetry may *increase* rather than decrease the fraction of supersymmetry preserved.

The possibility of rotating and boosting M-branes, while preserving 1/2 supersymmetry, is reflected in the Lorentz invariance of the effective worldvolume action. One might therefore expect the possibility of ‘mixed’ M-branes to be reflected in the structure of the M-5-brane effective action. We shall confirm this expectation by showing that the supertranslation algebra of the M-5-brane acquires a central extension with charges given precisely by the rotationally-invariant generalization of (8). In the M-2-brane case the 2-form central charge arises as a consequence of the existence of a ‘Wess-Zumino’ (WZ) term in the effective action [12]. It was assumed in [2] that the 5-form charge in the supertranslation algebra would have a similar justification, and this was made more plausible by some early partial results on the form of the WZ term [13], but it is only now that the κ -symmetric M-5-brane action has been found [5, 6] that the central charge structure can be definitively ascertained.

In carrying this out we have found a number of unexpected features that are absent from the 2-brane case. Firstly, it is not only Y that appears as a central charge in the M-5-brane’s supertranslation algebra, but also Z . This is not altogether unexpected in view of the fact that the M-5-brane can carry the 2-form charge Z , in addition to Y . More surprising is the fact that the WZ term in the M-5-brane action is not exclusively responsible for this charge. This is due to the fact, which we explain, that the underlying ‘naive’ supersymmetry algebra is not the standard one but is rather the M-2-brane algebra, which already has a 2-form central charge. Taking into account *all* sources of central charges in the supertranslation algebra of the M-5-brane we find (not unexpectedly) that they are just what is required for preservation of 1/2 supersymmetry.

As is well-known, preservation of 1/2 supersymmetry by an extended object is directly related to κ -symmetry of its effective worldvolume action, so the κ -symmetry transformations of the M-5-brane must encode the information about the M-theory supertranslation algebra. Indeed, we shall show that the central charge structure of the M-5-brane’s supertranslation algebra can be deduced directly from

the projector occurring in the proof of κ -symmetry. Related observations on the connection between supersymmetry and the κ -symmetry projector have been made previously in [14]. Also, the central extensions in the algebra of fermionic constraints (or supercovariant derivatives) have been discussed in [15] but, for reasons explained in [16], there is not generally a simple relation between the constraint algebra and the algebra of supersymmetry Noether charges considered here.

2 Superspace preliminaries

We begin by summarising the salient features of D=11 superspace. Let $Z^M = (X^m, \Theta^\alpha)$ be the coordinates of D=11 superspace. We may introduce a D=11 supervielbein E_M^A as the coordinate basis components of the frame 1-forms $E^A = (E^a, E^\alpha)$. In addition to the usual superspace 3-form gauge potential $C^{(3)}$ of D=11 supergravity we introduce a 6-form gauge potential $C^{(6)}$ [17]. The gauge-invariant field strengths are

$$\begin{aligned} R^{(4)} &\equiv dC^{(3)}, \\ R^{(7)} &\equiv dC^{(6)} - \frac{1}{2}C^{(3)}R^{(4)}, \end{aligned} \quad (9)$$

where the exterior product of forms is understood. The on-shell D=11 supergravity constraints imply, in particular, that these field strengths take the form

$$\begin{aligned} R^{(4)} &= \frac{i}{2}E^a E^b E^\alpha E^\beta (\Gamma_{ab})_{\alpha\beta} + \frac{1}{4!}E^a E^b E^c E^d F_{dcba}^{(4)} \\ R^{(7)} &= \frac{i}{5!}E^{a_1} \dots E^{a_5} E^\alpha E^\beta (\Gamma_{a_1 \dots a_5})_{\alpha\beta} + \frac{1}{7!}E^{a_1} \dots E^{a_7} F_{a_7 \dots a_1}^{(7)}, \end{aligned} \quad (10)$$

where the 7-form $F^{(7)}$ is the Hodge dual of the 4-form $F^{(4)}$. One solution to the full set of constraints is flat D=11 Minkowski spacetime with vanishing $F^{(4)}$; we shall call this the ‘D=11 vacuum’. The corresponding superspace admits the supertranslation group as a group of isometries. This includes the supersymmetry transformations

$$\delta\Theta = \epsilon, \quad \delta X^m = i\bar{\epsilon}\Gamma^m\Theta. \quad (11)$$

Note that in the D=11 vacuum we need not distinguish between frame indices ‘a’ and coordinate indices ‘m’; we shall use the coordinate indices in what follows.

In the D=11 vacuum, E^A are the left-invariant 1-forms on superspace, i.e. $E^A = (\Pi^m, d\Theta^\alpha)$ where

$$\Pi^m = dX^m + id\bar{\Theta}\Gamma^m\Theta. \quad (12)$$

In addition, the expressions (10) now simplify to

$$R^4 = \frac{i}{2}\Pi^m \Pi^n d\Theta^\alpha d\Theta^\beta (\Gamma_{mn})_{\alpha\beta}, \quad R^{(7)} = \frac{i}{5!}\Pi^{m_1} \dots \Pi^{m_5} d\Theta^\alpha d\Theta^\beta (\Gamma_{m_1 \dots m_5})_{\alpha\beta}. \quad (13)$$

These differential forms are clearly supertranslation invariant. Also, the 4-form $R^{(4)}$ and the 7-form $R^{(7)} + (1/2)C^{(3)}R^{(4)}$ are closed. These facts imply that the supersymmetry variations of $C^{(3)}$ and $C^{(6)}$ take the form

$$\begin{aligned} \delta C^{(3)} &= id[\bar{\epsilon}\Delta_2] \\ \delta C^{(6)} &= id[\bar{\epsilon}\Delta_5] + \frac{i}{2}(\bar{\epsilon}\Delta_2)R^{(4)}, \end{aligned} \quad (14)$$

where the spinor-valued p-form Δ_p ($p = 2, 5$) takes the form

$$\Delta_p = \frac{1}{p!}\Gamma_{m_1 \dots m_p}\Theta\Pi^{m_1} \dots \Pi^{m_p} + \dots \quad (15)$$

where ‘...’ indicates terms cubic or higher in Θ ; the full expression for Δ_2 can be found in [12]. We shall need the supersymmetry variations of Δ_2 and Δ_5 . These are determined by cohomological descent to be

$$\delta\Delta_2 = \mathcal{A}_2\epsilon, \quad (16)$$

$$\delta\Delta_5 = [\mathcal{A}_5 - \frac{1}{2}\mathcal{A}_2 C^{(3)}]\epsilon \quad (17)$$

where the matrix-valued p-forms \mathcal{A}_p have the form

$$\mathcal{A}_p = \frac{1}{p!}\Gamma_{m_1\dots m_p}dX^{m_1}\dots dX^{m_p} + d\Lambda_{(p-1)}, \quad (18)$$

The specific forms of Λ_{p-1} are not relevant here but can be found for $p = 2$ in [12]. Note that the entries of \mathcal{A}_p are p-forms which are closed but not exact, in an appropriate cohomology (de Rahm in the case of toroidally compactified space).

3 M-5-brane: κ -symmetry and supersymmetry

We must now review the structure of the manifestly $d = 6$ general coordinate invariant form [5] of the M-5-brane action.

Let ξ^i ($i = 0, 1, \dots, 5$) be the worldvolume coordinates of the fivebrane. The worldvolume fields comprise the maps $Z^M(\xi)$ from the worldvolume to superspace, and a 2-form gauge potential $A(\xi)$ with ‘modified’ field strength [18]

$$H_{ijk} = \partial_{[i}A_{jk]} - C_{ijk}^{(3)}, \quad (19)$$

where $C_{ijk}^{(3)}$ is the pullback of the superspace 3-form gauge potential $C^{(3)}$. In order that the M-5-brane action be invariant under super-isometries of the D=11 vacuum background we must require H to be supersymmetry invariant. In view of the supersymmetry transformation of $C^{(3)}$ we must set

$$\delta A = i\bar{\epsilon}\Delta_2, \quad (20)$$

where Δ_2 is here to be understood as the pullback to the worldvolume of the superspace 2-form introduced in (14). For the construction of the M-5-brane action, we also need the worldvolume six-form $C_{i_1\dots i_6}^{(6)}$ induced by the superspace 6-form gauge potential, and the induced worldvolume metric $g_{ij}(\xi) = E_i^a E_j^b \eta_{ab}$, where η is the D=11 Minkowski metric and $E_i^a = \partial_i Z^M E_M^a$.

The 3-form field strength H is required to be self-dual in a generalized sense, the self-duality condition being a consequence of the gauge field A equation of motion [19]. The manifestly $d = 6$ general coordinate invariance of the M-5-brane action is achieved if we introduce an auxiliary worldvolume scalar field $a(\xi)$ [20] (which is inert under super-isometries of the D=11 superspace background). Defining $g = \det(g_{ij})$, and

$$(H^*)^{ijk} = \frac{1}{3!\sqrt{-g}}\epsilon^{ijk i' j' k'} H_{i' j' k'}, \quad \tilde{H}^{ij} = \frac{1}{\sqrt{-(\partial a \cdot \partial a)}} (H^*)^{ijk} \partial_k a, \quad (21)$$

we can now write the M-5-brane action as

$$S = \int d^6\xi (L_0 + L_{WZ}), \quad (22)$$

where

$$L_0 = -\sqrt{-\det(g_{ij} + \tilde{H}_{ij})} + \frac{\sqrt{-g}}{4(\partial a \cdot \partial a)} (\partial_i a) (H^*)^{ijk} H_{jkl} (\partial^l a) \quad (23)$$

$$L_{WZ} = \frac{1}{6!}\epsilon^{i_1\dots i_6} [C_{i_1\dots i_6}^{(6)} + 10H_{i_1 i_2 i_3} C_{i_4 i_5 i_6}^{(3)}]. \quad (24)$$

The key feature of this action is its invariance under κ -symmetry transformations. On general grounds the variation $\delta_\kappa Z^M E_M^\alpha$ must take the form $[\mathcal{P}\kappa]^\alpha$ where $\kappa(\xi)$ is the D=11 spinor parameter and \mathcal{P} is a projector with $\text{tr}\mathcal{P} = 16$. Such a projector can be written as $\mathcal{P} = (1/2)(1 + \Gamma_*)$ where the matrix Γ_* is tracefree and squares to the identity. For the M-5-brane we have [5]

$$\Gamma_* = \frac{1}{\sqrt{\det(g_{ij} + \tilde{H}_{ij})}} \left[(\partial_i a \Gamma^i) \Gamma_j t^j - \frac{\sqrt{-g}}{2\sqrt{-(\partial a \cdot \partial a)}} (\partial_i a \Gamma^i) \Gamma^{jk} \tilde{H}_{jk} \right. \\ \left. + \frac{1}{5!(\partial a \cdot \partial a)} (\partial_i a \Gamma^i) \Gamma_{i_1 \dots i_5} \varepsilon^{i_1 \dots i_5 j} (\partial_j a) \right], \quad (25)$$

where (note that $t^i \partial_i a \equiv 0$)

$$t^i = \frac{1}{8(\partial a \cdot \partial a)} \varepsilon^{ij_1 j_2 k_1 k_2 \ell} \tilde{H}_{j_1 j_2} \tilde{H}_{k_3 k_4} \partial_\ell a \quad (26)$$

and $\Gamma_i = E_i^a \Gamma_a$ are the pullbacks to the worldvolume of the $D = 11$ Dirac matrices.

The matrix Γ_* has similar properties to the matrix $\bar{\Gamma}$ introduced previously in the discussion of the supertranslation algebra. It also has a similar structure, which is even more evident if in (25) we choose the temporal gauge

$$a(\xi) = t, \quad (27)$$

which is possible because of the invariance of the M-5-brane action under the local transformations $\delta a = \phi(\xi)$, $\delta A = \phi(\xi)f$, where f is a worldvolume 2-form constructed from H and a and given explicitly in [5]. In this gauge, and considering an infinite planar fivebrane in the D=11 vacuum, for which the induced metric is flat, we find that

$$\Gamma_* = \frac{1}{\sqrt{\det(\delta_{ij} + \tilde{H}_{ij})}} [\Gamma^0 \Gamma_{\underline{i}} t^{\underline{i}} - \frac{1}{2} \Gamma^0 \Gamma^{ij} \tilde{H}_{ij} + \frac{1}{5!} \Gamma^0 \Gamma_{\underline{i}_1 \dots \underline{i}_5} \varepsilon^{\underline{i}_1 \dots \underline{i}_5}]. \quad (28)$$

We may choose five of the ten space coordinates $X^{\underline{m}}$ to be the space coordinates $\sigma^{\underline{i}}$ ($\underline{i} = 1, \dots, 5$) of the fivebrane. If we then set

$$\tilde{H}_{ij} = -Z_{ij} \quad (29)$$

and take the time component of Z to vanish, as before, we find that $\Gamma_* = \bar{\Gamma}$ with

$$Y^{\underline{i}_1 \dots \underline{i}_5} = \varepsilon^{\underline{i}_1 \dots \underline{i}_5} \\ P^{\underline{i}} = \frac{1}{8} \varepsilon^{\underline{i} \underline{j}_1 \underline{j}_2 \underline{j}_3 \underline{j}_4} Z_{\underline{j}_1 \underline{j}_2} Z_{\underline{j}_3 \underline{j}_4} \\ P^0 = \sqrt{\det(\delta_{\underline{i} \underline{j}} + Z_{\underline{i} \underline{j}})}, \quad (30)$$

all other components of the charges vanishing. The construction guarantees that this charge configuration preserves 1/2 supersymmetry but this can be verified directly by use the following identity satisfied by any antisymmetric 6×6 matrix Z :

$$\det(1 + Z) = 1 - \frac{1}{2} \text{tr} Z^2 + \frac{1}{8} (\text{tr} Z^2)^2 - \frac{1}{4} \text{tr} Z^4. \quad (31)$$

By means of an $SO(5)$ transformation we can bring the 2-form charge Z to a form in which it has only two independent non-zero components $Z_{12} = -Z_{21} = \zeta_1$, $Z_{34} = -Z_{43} = \zeta_2$. This yields the charge configuration of (8). Thus, what we have found here from κ -symmetry considerations is the $SO(5)$ invariant generalization of the charge configuration (8), derived there from the requirement of preservation of 1/2 supersymmetry.

In view of the well-known connection between the preservation of 1/2 supersymmetry by extended objects and κ -symmetry of their effective actions, it is not too surprising that the κ -symmetry projector encodes the form of the central charges in the M-5-brane superalgebra that ensures preservation of 1/2

supersymmetry. As things stand, however, this connection is no more than an observation that two matrices happen to coincide if the variables in one are related to those in the other in a particular way. To show that this is no mere coincidence we must compute the central charge structure of the M-5-brane superalgebra and verify that the charges are given by (29) and (30).

4 M-5-brane superalgebra

We have seen that the M-5-brane Lagrangian L can be written as $L = L_0 + L_{WZ}$. In the D=11 vacuum the Lagrangian L_0 is invariant under the global supersymmetry transformations

$$\delta\Theta = \epsilon, \quad \delta X^m = i\bar{\epsilon}\Gamma^m\Theta, \quad \delta A = i\bar{\epsilon}\Delta_2. \quad (32)$$

The anticommutators of the corresponding Noether charges, computed via the canonical (anti)commutation relations of the worldvolume fields, yield what we will call the ‘naive’ supertranslation algebra. The form of this algebra is not specific to L_0 ; we would get the same result for any Lagrangian invariant under the supersymmetry transformations (32). The reason for the terminology ‘naive’ is that the true supersymmetry algebra of the M-5-brane will be a central extension of the naive one as a consequence of the fact that the WZ term L_{WZ} is not invariant under the transformations (32) but rather, as we shall see, changes by a total derivative.

One might suppose that the ‘naive’ supersymmetry algebra is just the standard one, as is the case for the usual form of the κ -symmetric M-2-brane action. But this is not so for the M-5-brane. The commutator of two supersymmetry transformations acting on A is

$$\{Q, Q\}A = \mathcal{A}_2, \quad (33)$$

where \mathcal{A}_2 (eq. (18)) is now to be understood as the pullback to the world volume of the matrix valued 2-form found from the supersymmetry variation of Δ_2 . Thus,

$$\{Q, Q\}A_{ij} = \partial_i X^m \partial_j X^n \Gamma_{mn} + \partial_{[i} \Lambda_{j]}. \quad (34)$$

When this is integrated over a 2-cycle \mathcal{M}_2 in the fivebrane we find

$$\{Q, Q\} \int_{\mathcal{M}_2} A = \frac{1}{2} \Gamma_{mn} \int_{\mathcal{M}_2} dX^m dX^n, \quad (35)$$

but the right hand side is just the 2-form central charge occurring in the M-2-brane supertranslation algebra.

In the Hamiltonian formulation the supersymmetry charge $Q_\alpha^{(0)}$ which generates (32)–(35) is expressed as an integral over the fivebrane at fixed time, \mathcal{M}_5 , as follows

$$Q_\alpha^{(0)} = i \int d^5\sigma [(\pi + i\Theta\Gamma^m\mathcal{P}_m) + i\mathcal{P}^{\underline{i}\underline{j}}(\Delta_2)_{\underline{i}\underline{j}}], \quad (36)$$

where π , \mathcal{P}_m and $\mathcal{P}^{\underline{i}\underline{j}}$ are the variables canonically conjugate to Θ , X^m and $A_{\underline{i}\underline{j}}$, respectively, derived from the full M-5-brane Lagrangian $L_0 + L_{WZ}$ (we shall give the explicit form of \mathcal{P}_m and $\mathcal{P}^{\underline{i}\underline{j}}$ below). Using the canonical quantum (anti)commutation relations we find that

$$\{Q_\alpha^{(0)}, Q_\beta^{(0)}\} = (C\Gamma^m)_{\alpha\beta} P_m + \frac{1}{2} (C\Gamma_{mn})_{\alpha\beta} Z_0^{mn}, \quad (37)$$

where P_m is the integral over \mathcal{M}_5 of the density \mathcal{P}_m , and

$$Z_0^{mn} = - \int_{\mathcal{M}_5} dX^m dX^n \mathcal{P}^*, \quad (38)$$

with \mathcal{P}^* the 3-form dual of \mathcal{P} , i.e. $\mathcal{P}_{i_1 i_2 i_3}^* = \frac{1}{2} \varepsilon_{i_1 i_2 i_3 i_4 i_5} \mathcal{P}^{i_4 i_5}$.

We conclude that the ‘naive’ supertranslation algebra, i.e. the algebra of Noether charges that would be associated with an invariant fivebrane Lagrangian, already includes a 2-form central charge! In fact, there is a sense in which this is already true for the M-2-brane. In the ‘scale-invariant’ formulation of the M-2-brane action [21] the WZ term is replaced by a two-form gauge potential with essentially the same ‘modified’ field strength tensor as that of the M-5-brane 2-form field A (the only difference is the dimension of the worldvolume on which the 3-form field strength is defined). In this formulation of the M-2-brane the ‘naive’ algebra is a 2-form central extension of the standard supertranslation algebra. This leads us to expect that the supertranslation algebra of the complete M-5-brane action will also contain a 2-form charge proportional to the constant ‘expectation value’ of the 3-form H . As we shall see below there are actually two equal contributions of this type; one is the ‘naive’ contribution under discussion here while the other arises from the non-invariance of the WZ term.

So we turn now to the WZ Lagrangian. It is convenient to rewrite it in differential form notation as

$$L_{WZ} = C^{(6)} + \frac{1}{2} H \wedge C^{(3)}. \quad (39)$$

Its supersymmetry variation is

$$\delta L_{WZ} = id(\bar{\varepsilon} \Delta) \quad (\Delta \equiv \Delta_5 - \frac{1}{2} \Delta_2 H), \quad (40)$$

where the p-forms Δ_p are to be understood as pullbacks to the worldvolume of the corresponding super-space p-forms defined in (16), so Δ is a worldvolume 5-form. We see that the supersymmetry variation of the M-5-brane action includes a boundary term. This term contributes to the supercharge, which has the form

$$\begin{aligned} Q_\alpha &= Q_\alpha^{(0)} + \int_{\mathcal{M}_5} \Delta_\alpha \\ &= i \int d^5 \sigma \left[(\pi + i \bar{\Theta} \Gamma^m \mathcal{P}_m)_\alpha + i (\mathcal{P}^{i_1 i_2} + \frac{1}{4} H^{*0 i_1 i_2}) (\Delta_{i_1 i_2}^2)_\alpha - i \varepsilon^{i_1 \dots i_5} (\Delta_{i_1 \dots i_5}^5)_\alpha \right]. \end{aligned} \quad (41)$$

Taking the (quantum) anticommutator of these supercharges we find that

$$\{Q_\alpha, Q_\beta\} = (C \Gamma^m)_{\alpha\beta} P_m + \frac{1}{2} (C \Gamma_{mn})_{\alpha\beta} Z^{mn} + \frac{1}{5!} \Gamma_{m_1 \dots m_5} Y^{m_1 \dots m_5}, \quad (42)$$

where

$$Y^{m_1 \dots m_5} = \int_{\mathcal{M}_5} dX^{m_1} \dots dX^{m_5}, \quad (43)$$

while the 2-form central charge is now the sum $Z = Z_0 + Z_{WZ}$, where Z_0 is given by (38) and

$$Z_{WZ}^{mn} = -\frac{1}{2} \int_{\mathcal{M}_5} dX^m dX^n (H + C^{(3)}). \quad (44)$$

The $C^{(3)}$ term in this expression is due to the $C^{(3)}$ term in (17).

To compute Z_0 we choose the temporal gauge (27), then:

$$\mathcal{P}^{i j} \equiv \frac{1}{\sqrt{-g}} \frac{\delta L}{\delta (\partial_0 A_{i j})} = \frac{1}{4} (H^{*0 i j} + C^{*0 i j}). \quad (45)$$

Equivalently $\mathcal{P}^* = (1/2)(H + C^{(3)})$, so $Z_0 = Z_{WZ}$. Since $H + C^{(3)} = dA$ we conclude² that

$$Z^{mn} = - \int_{\mathcal{M}_5} dX^m dX^n dA. \quad (46)$$

²It also follows that the expression (45) for \mathcal{P} does not contain time derivatives and is therefore a constraint. This constraint reflects the self-duality of the worldvolume field strength H .

Like Y (and in accord with general principles [16]), this is a topological charge. It is non-zero only for topologically nontrivial configurations of the self-dual field; for instance, when dA is a constant 3-form. The 5-form charge Y is just the electric source of $C^{(6)}$ or, equivalently, the magnetic source of $C^{(3)}$. The 2-form charge Z is the electric source of $C^{(3)}$, as is easily seen by variation of the M-5-brane action with respect to this background field which couples to the fivebrane via the ‘modified’ 3-form field strength H . What we have now shown is that both these M-brane charges appear as central charges in the M-5-brane supertranslation algebra.

If we now consider an infinite planar fivebrane in the D=11 vacuum and choose five of the ten space coordinates $X^{\underline{m}}$ to coincide with the five space coordinates $\sigma^{\underline{i}}$ of the fivebrane then the central charges Z and Y can be written as

$$Y^{\underline{i}_1 \dots \underline{i}_5} = \varepsilon^{\underline{i}_1 \dots \underline{i}_5} \quad Z^{\underline{i}\underline{j}} = -\tilde{H}^{\underline{i}\underline{j}}, \quad (47)$$

with all other components vanishing, and where \tilde{H} is now to be understood as a constant ‘expectation value’ of the worldvolume field. Note that $C^{(3)} = 0$ and the induced metric is flat in this case, so $\tilde{H}_{\underline{i}\underline{j}} = \tilde{H}^{\underline{i}\underline{j}}$.

To complete the determination of the $\{Q, Q\}$ anticommutator we must compute the ‘non-anomalous’ term proportional to P_m . In the context of the M-5-brane action P_m is just the integral over the fivebrane of the variable conjugate to X^m :

$$P_m = \int d^5\sigma \frac{\delta L}{\delta(\partial_t X^m)}. \quad (48)$$

Explicit computation leads to the conclusion that the only non-zero components of P_m are P_0 and $P_{\underline{i}}$ (the components of $P_{\underline{m}}$ parallel to the fivebrane). Using the relation in (47) between H and the 2-form charge Z we then find (for an infinite planar fivebrane in the D=11 vacuum with a flat induced metric)

$$P^0 = \sqrt{\det(\delta_{\underline{i}\underline{j}} + Z_{\underline{i}\underline{j}})}, \quad P^{\underline{i}} = \frac{1}{8} \varepsilon^{\underline{i}\underline{j}_1 \underline{j}_2 \underline{j}_3 \underline{j}_4} Z_{\underline{j}_1 \underline{j}_2} Z_{\underline{j}_3 \underline{j}_4}. \quad (49)$$

Of course, P^0 must be interpreted as the fivebrane tension. The fact that the membrane charge contributes to the M-5-brane tension is not surprising. The fact that the momentum is generally forced to be non-zero is somewhat surprising; it is a consequence of the term in L_0 quadratic in H .

We have now determined the full supertranslation algebra of the M-5-brane. It has a 2-form and a 5-form central charge given by (46), (43) and the 11-momentum given by (49). These are precisely the results of (29) and (30) anticipated earlier by consideration of κ -symmetry; as we saw there, these charges are just such as to ensure the preservation of 1/2 supersymmetry.

5 Comments

We have seen that the M-5-brane supertranslation algebra allows the possibility of ‘mixed’ M-brane configurations preserving 1/2 supersymmetry. In the absence of KK-branes the time components of the 2-form Z and the 5-form Y vanish. The spatial components of these charges in directions orthogonal to the fivebrane also vanish. The remaining non-zero components can be brought, by an $SO(5)$ transformation, to the form (8) in which ζ_1 and ζ_2 can be interpreted as the charges associated with two overlapping M-2-branes stretched along orthogonal directions inside the fivebrane. The whole configuration of branes moves along the fifth direction of the fivebrane with the momentum $\zeta_1 \zeta_2$ (or one can say that a wave propagates along the fivebrane in this direction). A slight generalization is possible (still preserving 1/2 supersymmetry) in which P has a non-zero component orthogonal to the fivebrane, but this corresponds to a boost of the configuration just described.

For the special case in which $\zeta_2 = 0$ the associated 1/2 supersymmetric solution of D=11 supergravity is known [11]. The D=11 supergravity solution corresponding to the more general case, which will be stationary rather than static, is not yet known. It seems likely that it could be constructed as the

lift to D=11 of a U-dualized extreme black hole solution in D=6. Given such a solution, it could be dimensionally reduced to a static solution of IIA supergravity, in which context it could be interpreted as a non-marginal bound state in IIA superstring theory of a D-0-brane at the intersection of two D-2-branes within a D-4-brane. This type of ‘mixed’ D-brane configuration has been discussed previously by various authors, e.g. [22]. These possibilities could of course be deduced directly from the central charge structure of the D-p-brane supertranslation algebras, but the results so obtained must be related by dualities to those obtained here for the M-5-brane.

Returning to the M-theory algebra in the form (3) we can ask whether there are any further possibilities with $\bar{\Gamma}^2 = 1$ that could have a ‘mixed’ M-brane interpretation. We have found only one example (excluding KK charges). This is when P , Z and Y each have spatial components that are completely orthogonal to each other (for instance, P_1 , Z_{23} and Y_{45678}). Since, in this case, non-zero momentum corresponds to an orthogonal boost we can set it to zero without loss of generality. The resulting Z and Y charges could, in principle, correspond to a non-marginal bound state of an M-2-brane orthogonally intersecting an M-5-brane. If there were such a bound state then duality would imply the existence of a non-marginal 0-brane/6-brane bound state (preserving 1/2 supersymmetry) in IIA superstring theory, but no such bound state exists because the force between the constituents is repulsive rather than attractive [23]. We thus conclude that *all non-marginal bound states of M-branes preserving 1/2 supersymmetry are accounted for by the M-5-brane effective action*. Since M-wave and M-2-brane are in the same equivalence class as the M-5-brane under duality (at least after compactification on T^2) we see that M-theory is essentially the theory of a single object, but one which takes on various forms in various dual formulations of the theory.

One surprising result of our analysis is that the 2-form central extension in the M-5-brane algebra is not entirely due to the WZ term in the action. The source of the other contribution is reminiscent of the source of the 2-form central charge in the ‘scale-invariant’ formulation [21] of the M-2-brane action. Together, these facts suggest that the realization of supersymmetry as translations in superspace is not the ideal way to think about it. Instead, one needs to consider something else, perhaps a free-differential algebra, in which the occurrence of the 2-form charge in the supertranslation algebra is automatic.

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